Assignment 1: Building Mental Muscle for giving Correctness Argument!

Direct proof, contraposition, contradiction, invariants, and mathematical evidence

Instructions. Answer every question with complete reasoning. When both a mathematical and an intuitive argument are requested, give them separately. Examples may suggest an idea, but a general claim requires a proof. The hints indicate a route; they are not a substitute for your argument.

1. A Pattern, a Program, and a Counterexample [Computation vs. proof]

Write a program that evaluates $P(n) = n^2 + n + 41$ for $n = 0, 1, 2, \ldots$ and stops at the first composite output. Print the input, output, and a nontrivial factorization. Then explain why many prime outputs do not prove that $P(n)$ is always prime, while one composite output is enough to disprove that claim.

Directions to think:
1. Build a reliable prime test that checks possible divisors only up to $\sqrt{P(n)}$.
2. Search inputs in increasing order if you want to claim that a failure is the first one.
3. Contrast a finite list of successful tests with a statement about infinitely many integers.

2. Consecutive Integers [Direct proof]

Prove that the product of any two consecutive integers is even. First give a formal algebraic proof. Then give an intuitive explanation based on what must be true of one number in every consecutive pair.

Directions to think:
1. Represent the pair as n and $n + 1$.
2. Consider the two cases: n is even or n is odd.
3. For the intuitive argument, think about how parity alternates along the number line.

3. A Divisibility Structure [Direct proof]

Let $a, b \in \mathbb{Z}$. Prove directly: if $6 \mid a$ and $15 \mid b$, then $3 \mid (a + b)$. State where you use the definitions of divisibility and closure of the integers.

Directions to think:
1. Translate the assumptions into $a = 6r$ and $b = 15s$ for integers r, s.
2. Add the expressions and factor out the divisor required in the conclusion.
3. Check that the remaining expression is an integer before concluding divisibility.

4. Squares and Multiples of Three [Proof by contraposition]

Prove: if n^2 is divisible by 3, then n is divisible by 3. Use contraposition, and also explain the result intuitively using the possible remainders after division by 3.

Directions to think:
1. The contrapositive begins: if $3 \nmid n$, then $3 \nmid n^2$.
2. If $3 \nmid n$, then n has remainder 1 or 2 modulo 3.
3. Square both possible remainders and inspect the new remainder.

5. A Real-Life Contrapositive [Logical reasoning]

A building policy states: "If the laboratory is occupied, then the ventilation system is running." Write its converse and contrapositive. Explain which statement allows a safety inspector to conclude that the laboratory is empty after observing that the ventilation system is not running. Also explain why observing ventilation alone does not prove that the laboratory is occupied.

Directions to think:
1. Identify $P =$ "occupied" and $Q =$ "ventilation running."
2. The contrapositive reverses and negates: $\neg Q \to \neg P$.
3. Do not confuse the converse $Q \to P$ with a statement logically equivalent to the policy.

6. Irrationality of $\sqrt{3}$ [Proof by contradiction]

Prove that $\sqrt{3}$ is irrational. You may use the fact that $3 \mid m^2$ implies $3 \mid m$. After the formal proof, explain intuitively why showing that both numerator and denominator share a factor destroys the claim that a fraction was in lowest terms.

Directions to think:
1. Assume $\sqrt{3} = m/n$, where m/n is in lowest terms and $n \neq 0$.
2. Square the equation, show $3 \mid m$, write $m = 3k$, and substitute back.
3. Show that $3 \mid n$ as well, then identify the contradiction.

7. The Chef's Black-and-White Bean Jar [Termination and invariant]

A jar initially contains 75 black beans and 150 white beans. The chef repeatedly draws two beans and follows these rules:

- If both beans are black, the chef removes both and puts one black bean into the jar.
- If at least one drawn bean is white, the chef removes one white bean and returns the other drawn bean to the jar.

The choices of beans may be completely arbitrary.

(a) Prove that the process must end after finitely many steps. Determine the exact number of steps.
(b) Find a property involving black beans that remains true after every move.
(c) Use that invariant to prove the color of the final bean, independent of the chef's choices.
(d) Give an intuitive explanation: why can the chef change the order of the moves but not the final answer?

Directions to think:
1. Track the total number of beans before and after every move.
2. Ask whether a move can eliminate the last remaining black bean. Check both rules.
3. When only one bean remains, combine the invariant with the possible final colors.

8. Rational Plus Irrational [Proof by contradiction]

Let r be rational and x irrational. Prove that $r + x$ is irrational. Then give a number-line intuition: explain why translating an irrational point by a rational distance cannot place it on a rational point.

Directions to think:
1. Assume for contradiction that $r + x$ is rational.
2. Rearrange to write $x = (r + x) - r$.
3. Use closure of rational numbers under subtraction and compare with the premise.

9. Choosing the Best Proof Method [Proof strategy]

For each claim below, choose *direct proof*, *contraposition*, or *contradiction*. Briefly justify the choice, then prove the claim.

(a) The sum of two odd integers is even.
(b) If n^2 is odd, then n is odd.
(c) There is no greatest integer.

Directions to think:
1. Definitions of even and odd make algebraic direct proofs natural.
2. For an implication about n^2, compare the direct route with the contrapositive.
3. To reject a "greatest" object, assume it exists and construct a larger one.

10. Hacker's Challenge: A Proof Behind an Algorithm [Algorithmic proof]

A primality tester tries divisors only from 2 through $\lfloor \sqrt{N} \rfloor$. Prove that if none divides $N > 1$, then N is prime. Give both a formal contradiction argument and an intuitive explanation using factor pairs.

Directions to think:
1. Assume N is composite and write $N = ab$ with $a, b > 1$.
2. If both factors exceeded $\sqrt{N}$, compare ab with N.
3. Explain why factors beyond $\sqrt{N}$ merely repeat partners already encountered below it.

Testing finds evidence. A proof explains why the conclusion must hold.