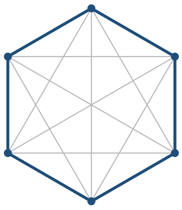
Assignment 2: Mathematical Induction

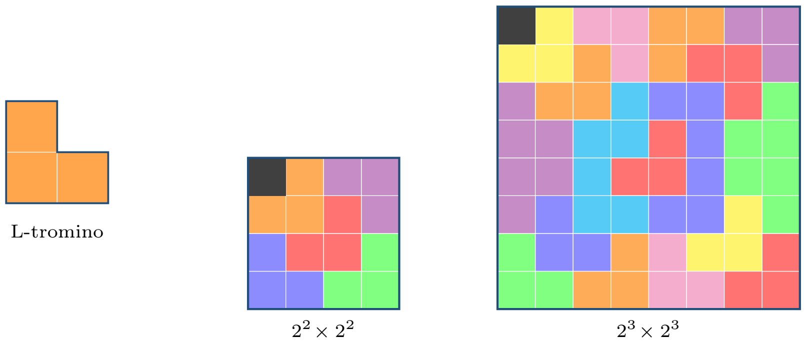
Algebraic identities and real-world applications

Problems	15	**Total marks**	75
Requirement	Answer all problems	**Method**	Mathematical induction

1. **Sum of Natural Numbers.** Prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for every integer $n \geq 1$. [5]
2. **Sum of Odd Numbers.** Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for every integer $n \geq 1$. [5]
3. **Sum of Squares.** Prove that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for every integer $n \geq 1$. [5]
4. **Geometric Series.** Prove that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for every integer $n \geq 0$. [5]
5. **Divisibility.** Prove that $7^n - 1$ is divisible by 6 for every integer $n \geq 1$. [5]
6. **An Inequality.** Prove that $2^n \geq n + 1$ for every integer $n \geq 0$. [5]
7. **Factorial Inequality.** Prove that $n! > 2^n$ for every integer $n \geq 4$. [5]
8. **Fibonacci Identity.** Let $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. Prove that $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$ for every $n \geq 1$. [5]
9. **Diagonals of a Convex Polygon.** A diagonal joins two non-adjacent vertices. Prove by induction that a convex polygon with $n \geq 3$ vertices has $\frac{n(n-3)}{2}$ diagonals. The sample hexagon has 9 diagonals. [5]

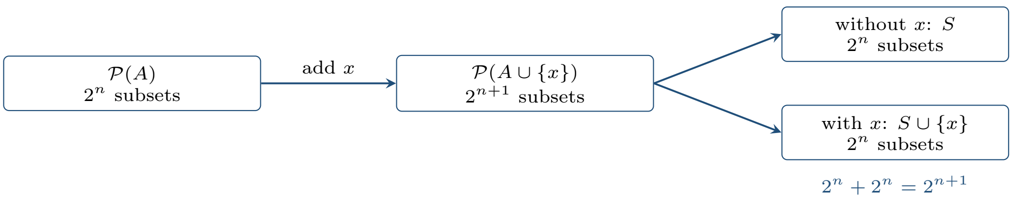


10. **Chocolate Bar.** A bar contains n unit squares. Each move breaks one existing piece into exactly two pieces. Prove that separating all squares requires exactly $n - 1$ moves. [5]
11. **Postage Stamps.** A post office sells only 4-cent and 5-cent stamps. Prove by strong induction that every postage amount of at least 12 cents can be formed. [5]
12. **Tower of Hanoi.** The puzzle has n disks and three pegs. Only one disk may move at a time, and a larger disk may not be placed on a smaller one. Prove that the minimum number of moves is $2^n - 1$. [5]
13. **Defective Chessboard.** One square is removed from a $2^n \times 2^n$ board. Prove that the remaining board can be tiled by L-shaped trominoes, each covering three squares. [5]



The black square is removed; each color represents one L-tromino.

14. **Two Party Lines.** Suppose n people, $p_1, \ldots, p_n$, attend a party. For every pair of people, either they shook hands during the party or they did not. Prove by induction that the people can always be arranged into two lines such that, in the first line, each person shook hands with the person immediately behind them, while in the second line, each person did not shake hands with the person immediately behind them. Either line may be empty. [5]
15. **Number of Subsets.** Prove by induction that an n-element set has exactly 2^n subsets. Use the picture below to explain the inductive step: after adding a new element x, the subsets split into those that do not contain x and those that do. [5]



For every proof, state $P(n)$ and label the base case, hypothesis, step, and conclusion.